Games, Auctions, Learning, and the Price of Anarchy

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Games and Quality of Solutions

• Rational selfish action can lead to outcome bad for everyone

Model:
• Value for each cow decreasing function of # of cows
• Too many cows: no value left

Tragedy of the Commons
Example: Routing Games

- Traffic subject to congestion delays
- cars and packets follow shortest path

Congestion game = cost (delay) depends only on congestion on edges
What is Selfish Outcome?

We will use: **Nash equilibrium**

- Current strategy “best response” for all players (no incentive to deviate)

**Theorem [Nash 1952]:**

- Always exists if we allow randomized strategies
Results for non-atomic games

Theorem 3 (Roughgarden-Tardos’02):

• In any network with continuous, nondecreasing latency functions

\[
\text{cost of Nash with rates } r_i \quad \leq \quad \text{cost of opt with rates } 2r_i \quad \text{for all } i
\]
Today: Online Markets

Advertisement  Effort  Information

Online markets use simple auctions for allocations

How good is the resulting allocation?
Ideal Auction

Basic Auction: single item Vickrey Auction

Player utility $v_i - p_i$ — item value –price paid

Vickrey Auction (second price) – Truthful – Efficient – Simple

Pays $5
Some Simple Auctions

First/Second price
GSP, etc

All Pay,
War of attrition

Most not truthful
## Truthful or Simple

<table>
<thead>
<tr>
<th>Vickrey, Clarke, Groves (VCG) truthful, but not simple</th>
<th>Online ads (Display/Search) customized by information about user: Search term, History of user, Time of the day, Geographic Data, Cookies, Budget</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Assignment efficient (maximizing social welfare)</td>
<td>Millions of ads each minute and all different!</td>
</tr>
<tr>
<td>• Payment welfare loss to others</td>
<td>Needs a simple and intuitive scheme</td>
</tr>
</tbody>
</table>
Should Work in Composition

- eBay
- Sotheby’s BIDnow
- Goods
- Amazon Mechanical Turk
- Effort
- Exelate
- Bluekai
- Information
- TopCoder
- Advertisement
- Google AdWords
Multiple Simple Auctions?

Second price auction

truthful and simple, but...

Two simultaneous second price auctions?  

Not

How about sequentially?  

Not

repeat offer to same or different seller
Goal [Syrkanis-Tardos 2013]: Design mechanisms such that a market composed of such mechanisms is approximately efficient?

Local: local mechanism efficient

Global: mechanism efficient
Today Mixing Auction Types

First/Second price/All pay/ etc.

Key idea: Auctions that price
First Price Auction: Good prices

Outcome of first price auction:
Each player $i$ has a bid $b'_i$, such that if current bids are $b_i$ and prices are $p_j$ we get

$$utility_i(b'_i, b_{-i}) \geq v_i(S_i^*) - \sum\{x \in S_i^*\} p_x$$

$\Rightarrow$ Pure Nash sets market clearing prices (Bikhchandani’96)
What are Good Prices?

Market clearing: \( utility_i(b) \geq v_i(S_i^*) - \sum_{x \in S_i^*} p_x \)
\[ \forall S_i^* \text{ and } \forall i \]

**Theorem:** Market clearing prices guarantee socially optimal outcome: maximizing \( \sum_i v_i(S_i) \) of all allocations \((S_1, S_2, \ldots)\)

**Proof:** Each player has value \( \geq \) favorite items

\[
\sum_i (v_i(S_i) - \sum_{x \in S_i} p_x) \geq \sum_i (v_i(S_i^*) - \sum_{x \in S_i^*} p_x)
\]
Robust solution concepts

- Pure Nash of Complete Information is very brittle
  - Pure Nash might not always exist
  - Game might be played repeatedly, with players using learning algorithms (correlated behavior)
  - Players might not know other valuations
  - Players might have probabilistic beliefs about values of opponents
What is Selfish Outcome (2)?

Do players find Nash?
   if solution stable that is Nash

But…..

Finding Nash is hard algorithmic problem
   (Daskalakis-Goldberg-Papadimitrou’06)

No regret learning: do at least as well as any fixed strategy with hindsight.

If converges: Nash equilibrium…
Run Auction on
( $b_1^1, b_2^1, ..., b_n^1$)

Run Auction on
( $b_1^t, b_2^t, ..., b_n^t$)

Maybe here they don’t know how to bid, who are the other players, ...

By here they have a better idea...

Vanishingly small regret for any fixed strat $b'$:

$$\sum_t \text{utility}_i(b_i^t, b_{-i}^t) \geq \sum_t \text{utility}_i(b', b_{-i}^t) - o(T)$$

Simple randomized strategies guarantee vanishing regret
(regret matching, multiplicative weight)
Bayesian Beliefs

Bayes-Nash Equilibrium:

\[ E_{v_{-i}}[utility_i(b(v))] \geq E_{v_{-i}}[utility_i(b'_i, b_{-i}(v_{-i}))] \]
Direct extensions

• What if conclusions drawn for the Pure Nash equilibrium of the complete information setting could be directly extended to these robust notions?

• Possible, but we need to restrict the type of analysis
Market Clearing Prices

**Market clearing:** Player $i$ has a bid $b'_i$ that guarantees

$$\text{utility}_i(b'_i, b_{-i}) \geq v_i(S^*_i) - \sum_{x \in S^*_i} p_x$$

**Robust:** bid $b'_i$ should not depend on $b_{-i}$ and other players.

Do robust bids ever exist?
Example: Approximately market clearing mechanism

Claim:
First price auction for a single item is \((\frac{1}{2}, 1)\) smooth

User of value \(v_i\) bid \(b'_i = \frac{1}{2} v_i\), utility

Claim: \(utility_i(b'_i, b_{-i}) \geq \frac{1}{2} v_i - 1 p_i\)

Proof

• Either wins and has utility \(v_i - p_i = \frac{1}{2} v_i\)

• Or looses and hence price was \(p_i \geq \frac{1}{2} v_i\)
Examples of approximately market clearing auction games

- First price auction \((1-1/e,1)\) approx
  - See also Hassidim et al EC’12, Syrkhanis’12
- All pay auction \((\frac{1}{2},1)\)-smooth
- First position auction (GFP) is \((\frac{1}{2},1)\)-smooth
- Second price auction is \((\frac{1}{2},0,1)\)-smooth (no overbidding)
- Generalized second price (GSP) is \((\frac{1}{2},0,1)\)-smooth

Other applications include:
- public goods
- bandwidth allocation (Johari-Tsitsiklis),
- etc
Auctions with OK Prices: Smooth

Approximately market clearing: Player $i$ has a bid $b'_i$, such that if current bids are $b_{-i}$ and item prices are $p_j$ we get $utility_i(b'_i, b_{-i}) \geq \lambda v_i(S_i^*) - \mu \sum_{x \in S_i^*} p_x$.

Or just $\sum_i utility_i(., b_{-i}) \geq \lambda \text{OPT} - \mu \sum_x p_x$

$b'_i$, should not depend on $b_{-i}$

Smooth games Roughgarden’09

Theorem Smooth mechanism at Nash has socially approximately optimal outcome (off by at most factor of $\lambda / \max(1, \mu)$)

Proof: at Nash $utility_i(b'_i, b_{-i}) \leq utility_i(b)$
Robust Price of Anarchy

Theorem (Syrkganis-T’13) Auction game \((\lambda,\mu)\)-smooth, then

- Price of anarchy is at most \(\max(1, \mu)/\lambda\)
- Preserved in Composition (assuming no complements)
- Also true for mixed equilibria (and even correlated equilibria = learning outcomes)
- Also true for games with uncertainty, assuming player types are independent

See in a bit
Robust Price of Anarchy

Theorem (Syrkganis-T’13) Auction game (\(\lambda,\mu\))-smooth game, then

- Price of anarchy is at most \(\max(1, \mu)/\lambda\)
- Also true for learning outcomes (= coarse correlated equilibria) and mixed equilibria

Proof: let \(b^1, b^2, \ldots, b^t, \ldots\) sequence of bids

\[
\sum_t \text{utility}_i(b^t) \geq \sum_t \text{utility}_i(b'_i, b^t_{-i}) \quad \text{(no regret)}
\]

\[
\sum_i \sum_t \text{utility}_i(b'_i, b^t_i) \geq \lambda T \text{ Opt} - \mu \sum_t \sum_x p^t_x \quad \text{(smooth)}
\]
Robust Price of Anarchy

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Simultaneous Composition

**Corollary:** Simultaneous first price auction has price of anarchy of $e/(e-1)$ if player values have no complements

- Simultaneous all-pay auction: price anarchy 2
- Mix of first price and all pay, price of anarchy $\leq 2$
No complements ⊇ Submodular

item auctions:

Submodular: Marginal value for any allocation can only decrease, by getting more items:

For all sets $S \subseteq S'$, and item $x \not\in S'$ we have

$$v(S + x) - v(S) \geq v(S' + x) - v(S')$$

Across Mechanisms (fractionally subadditive)
Simultaneous Composition

Theorem (Syrkganis-T’13): simultaneous item auctions where each is \((\lambda,\mu)\)-smooth and players have fractionally subadditive valuations, then composition is also \((\lambda,\mu)\)-smooth

Proof: valuation \(v(S)\) is fractionally subadditive \(\Rightarrow\) maximum of linear functions: \(v(S) = \max_j \sum_{x \in S} v^j_x\)
e.g., submodular

- optimal allocation \(S^*_1, S^*_2, \ldots\) values \(v_i(S^*_i) = \sum_{x \in S^*_i} v^{\{j_i\}}_x\)
- User \(i\) bids in auction \(x\) for value \(v^{\{j_i\}}_x\)
- Real value \(v_i(S) \geq \sum_{x \in S} v^{\{j_i\}}_x\)

Conclusion: Designing for Selfish Users

- Selfish users can ruin social welfare (Tragedy of the Commons)
- With care, we can design for selfish use and mitigate the welfare loss
- Guarantees very robust (Nash, learning, uncertainty)